Answers Chapter 1

Series Solution of Ordinary Differential Equations

Answer Exercise (1)

(1) Use power series to solve the equation $y'' + y = 0$

Answer:

We assume there is a solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$ (1)

We differentiate power series term by term, so

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad (2)$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \qquad (3)$$

In order to compare the expressions for and y'' more easily, we rewrite y'' as follows:"

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \qquad (4)$$

Substituting the expressions in Equations 2 and 4 into the differential equation, we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n\right] x^n = 0 \qquad (5)$$

If two power series are equal, then the corresponding coefficients must be equal. Therefore, the coefficients of x^n in Equation 5 must be 0:

$$(n+2)(n+1) a_{n+2} + a_n$$

$$a_{n+2} = \frac{-a_n}{(n+1)(n+2)}, \quad n = 0, 1, 2, 3, \ldots \qquad (6)$$

Equation (6) is called a recursion relation. If c_0, c_1 are known, this equation allows us to determine the remaining coefficients recursively by putting $n = 0, 1, 2, 3, \ldots$ in succession.

Put $n=0$ $\quad a_2 = \frac{-a_0}{1.2}$

Put $n=1$ $\quad a_3 = \frac{-a_1}{2.3}$

Put $n=2$ $\quad a_4 = \frac{-a_2}{3.4} = \frac{a_0}{1.2.3.4} = \frac{a_0}{4!}$

Put $n=3$ $\quad a_5 = \frac{-a_3}{4.5} = \frac{a_1}{2.3.4.5} = \frac{a_1}{5!}$

Put $n=4$ $\quad a_6 = \frac{-a_4}{5.6} = \frac{-a_0}{4!.5.6} = \frac{-a_0}{6!}.$

Put $n=5$ $\quad a_7 = \frac{-a_5}{6.7} = \frac{-a_1}{5!.6.7} = \frac{-a_1}{7!}$

By now we see the pattern:

For the even coefficient, $a_{2n} = (-1)^n \frac{a_0}{(2n)!}$

For the even coefficient, $a_{2n+1} = (-1)^n \frac{a_1}{(2n+1)!}$

Putting these values back into Equation 2, we write the solution as

$$= a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots\right)$$

$$+ a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots\right)$$

$$= a_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

---

(2) Use power series to solve the equation $y'' - 2xy' + y = 0$

Answer:

We assume there is a solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$

We can differentiate the power series term by term, so

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute in the equation

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2\sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Let the first series start from 0

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Separate first term from the first and third series to both start from 1

$$2a_2 + a_0 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=1}^{\infty} a_n x^n = 0$$

Now collect the series

$$2a_2 + a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - (2n-1)a_n\right] x^n = 0$$

Then we have

$$2a_2 + a_0 = 0 \rightarrow a_2 = -\frac{a_0}{2}$$

$$\left[(n+2)(n+1)a_{n+2} - (2n-1)a_n\right],\ n = 1,2,3,...$$

$$a_{n+2} = \frac{2n-1}{(n+1)(n+2)} a_n,\ n = 1,2,3,... \qquad (7)$$

We solve this recursion relation by putting successively in Equation 7

Put $n = 1$: $a_3 = \frac{1}{2.3} a_1$

Put $n = 2$: $a_4 = \frac{3}{3.4} a_2 = -\frac{3}{1.2.3.4} = \frac{3}{4!} a_0$

Put $n = 3$: $a_5 = \frac{5}{4.5} a_3 = \frac{5}{4.5}\frac{1}{2.3} a_1 = \frac{5}{5!} a_1$

Put $n = 4$: $a_6 = \frac{7}{5.6} a_4 = \frac{3.7}{5.6.4!} a_0 = -\frac{3.7}{6!} a_0$

Put $n = 5$: $a_7 = \frac{1.5.9}{7!} a_1$

Put $n = 6$: $a_8 = \frac{11}{7.8} a_6 = -\frac{3.7.11}{8!} a_0$

Put $n = 7$: $a_9 = \frac{13}{8.9} a_7 = -\frac{1.5.9.13}{9!} a_1$

In general, the even coefficients are given by $a_{2n} = -\dfrac{3.7.11...(4n-5)}{(2n)!}a_0$

And the odd coefficients are given by $a_{2n-1} = \dfrac{1.5.9...(4n-3)}{(2n+1)!}a_1$

The solution is

$$y = a_0\left(1 - \frac{1}{2!}x^2 - \frac{3}{4!}x^4 - \frac{3.7}{6!}x^6 - \frac{3.7.11}{8!}x^8 + ...\right)$$

$$+a_1\left(x + \frac{1}{3!}x^3 + \frac{1.5}{5!}x^5 + \frac{1.5.9}{7!}x^7 + \frac{1.5.9.13}{9!}x^9 + ...\right)$$

or

$$y = a_0\left(1 - \frac{1}{2!}x^2 - \sum_{n=2}^{\infty} \frac{3.7.11...(4n-5)}{(2n)!}x^{2n}\right)$$

$$+a_1\left(x + \sum_{n=1}^{\infty} \frac{1.5.9.13...(4n-3)}{(2n+1)!}x^{2n+1}\right). \qquad (8)$$

---

(3) Use power series to solve the differential equation. $y' - y = 0$

Answer:

Let $y = \sum_{n=0}^{\infty} c_n x^n$ Then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$

By substitute in the equation we have $\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$

Let the second series start from 1 as following $\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=1}^{\infty} c_{n-1} x^{n-1} = 0$

Now the power of x are equal and the two summation start from 1 then we write the equation in the form $\sum_{n=1}^{\infty} (nc_n - c_{n-1})x^{n-1} = 0$

$\therefore (nc_n - c_{n-1}) = 0 \qquad \Rightarrow c_n = \dfrac{c_{n-1}}{n}, \; n \ge 1$

$c_1 = c_0, \quad c_2 = \dfrac{1}{2}c_1 = \dfrac{c_0}{2.1}, \; c_3 = \dfrac{1}{3}c_2 = \dfrac{1}{3}.\dfrac{c_0}{2.1} = \dfrac{c_0}{3.2.1} = \dfrac{c_0}{3!}$ and $c_4 = \dfrac{c_0}{4!}$

The solution is $\boldsymbol{y = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{c_0}{n!} x^n = c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = c_0 e^x}$

---

(4) $y' - xy = 0$

Answer

Let $y = \sum_{n=0}^{\infty} c_n x^n$ Then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$

By substitute in the equation we have $\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$

Separate the first terms from the first series $c_1 + \sum_{n=2}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$

Let the first series start from 0 $c_1 + \sum_{n=0}^{\infty} (n+2) c_{n+2} x^{n+1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$

Collect the series $c_1 + \sum_{n=0}^{\infty} \left[(n+2) c_{n+2} - c_n\right] x^{n+1} = 0$

Compare the coefficient in both sides

$c_1 = 0$ and $(n+2)c_{n+2} - c_n$ then $c_{n+2} = \dfrac{c_n}{(n+2)},\ n \ge 0$

at $n = 0$ $c_2 = \dfrac{c_0}{2}$ at $n = 1$ $c_3 = \dfrac{c_1}{3} = 0$

at $n = 2$ $c_4 = \dfrac{c_2}{4} = \dfrac{c_0}{2.4}$ at $n = 3$ $c_5 = \dfrac{c_3}{5} = 0$

at $n = 4$ $c_6 = \dfrac{c_4}{6} = \dfrac{c_0}{2.4.6} = \dfrac{c_0}{2^3(1.2.3)} = \dfrac{c_0}{2^3.3!}$

$$c_{2n-2} = \frac{c_0}{2^{n-1}.(n-1)!},\ n \ge 2$$

Solution function is

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \dots + c_n x^n + \dots$$

$$= c_0 + \frac{c_0}{2} x^2 + \frac{c_0}{2.4} x^4 + \frac{c_0}{2.4.6} x^6 + \dots + \frac{c_0}{2^{n-1}.(n-1)!} x^{2n-2} + \dots$$

$$= c_0 \left[1 + \frac{1}{2} x^2 + \frac{1}{2.4} x^4 + \frac{1}{2.4.6} x^6 + \dots + \frac{1}{2^{n-1}.(n-1)!} x^{2n-2} + \dots\right]$$

$$= c_0 \sum_{n=1}^{\infty} \frac{x^{2n-2}}{2^{n-1}.(n-1)!}. = c_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n.(n)!}$$

(5) $y' - x^2 y = 0$

Answer

Let $y = \sum_{n=0}^{\infty} c_n x^n$ Then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$

By substitute in the equation we have $\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+2} = 0$

Separate the first two terms from the first series

$$c_1 + 2c_2 x + \sum_{n=3}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+2} = 0$$

Let the second series start from 3 then $c_1 + 2c_2 x + \sum_{n=3}^{\infty} n c_n x^{n-1} - \sum_{n=3}^{\infty} c_{n-3} x^{n-1} = 0$

Collect the series $c_1 + 2c_2 x + \sum_{n=3}^{\infty} [n c_n - c_{n-3}] x^{n-1} = 0$

Compare the coefficient in both sides

$c_1 = c_2 = 0$ and $n c_n - c_{n-3}$ then $c_n = \dfrac{c_{n-3}}{n},\ n \ge 3$

at $n = 3$ $c_3 = \dfrac{c_0}{3}$ at $n = 4$ $c_4 = \dfrac{c_1}{4} = 0$

at $n = 5$ $c_5 = \dfrac{c_2}{5} = 0$ at $n = 6$ $c_6 = \dfrac{c_3}{6} = \dfrac{c_0}{3.6}$

at $n = 7$ $c_7 = \dfrac{c_4}{7} = 0$ at $n = 8$ $c_8 = \dfrac{c_5}{8} = 0$

at $n = 9$ $c_9 = \dfrac{c_6}{9} = \dfrac{c_0}{3.6.9} = \dfrac{c_0}{3^3(1.2.3)} = \dfrac{c_0}{3^3.3!}$ $c_{3n} = \dfrac{c_0}{3^n.(n)!},\ n \ge 1$

Solution function is

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + ... + c_n x^n + ...$$

$$= c_0 + \frac{c_0}{3} x^3 + \frac{c_0}{3.6} x^6 + \frac{c_0}{3.6.9} x^9 + ... + \frac{c_0}{3^n.(n)!} x^{3n} + ...$$

$$= c_0 \left[1 + \frac{1}{3} x^3 + \frac{1}{3.6} x^6 + \frac{1}{3.6.9} x^9 + ... + \frac{c_0}{3^n.(n)!} x^{3n} + ... \right]$$

$$= c_0 \sum_{n=0}^{\infty} \frac{1}{3^n.(n)!} x^{3n}.$$

(6) $(x-3)y'+2y=0$

Answer

Let $y=\sum_{n=0}^{\infty} c_n x^n$ Then $y'=\sum_{n=1}^{\infty} n c_n x^{n-1}$

By substitute in the equation we have $\sum_{n=1}^{\infty} n c_n x^n - \sum_{n=1}^{\infty} 3n c_n x^{n-1} + \sum_{n=0}^{\infty} 2c_n x^n = 0$

Let the last series start from 1 then $\sum_{n=1}^{\infty} n c_n x^n - \sum_{n=1}^{\infty} 3n c_n x^{n-1} + \sum_{n=1}^{\infty} 2c_{n-1} x^{n-1} = 0$

Collect the series $\sum_{n=1}^{\infty} n c_n x^n - \sum_{n=1}^{\infty} \left(3n c_n - 2c_{n-1}\right) x^{n-1} = 0$

Separate the first term from the second series

$$\sum_{n=1}^{\infty} n c_n x^n - \left(3c_1 - 2c_0\right) - \sum_{n=2}^{\infty} \left(3n c_n - 2c_{n-1}\right) x^{n-1} = 0$$

Let the first series start from 2

$$\sum_{n=2}^{\infty} (n-1) c_{n-1} x^{n-1} - \left(3c_1 - 2c_0\right) - \sum_{n=2}^{\infty} \left(3n c_n - 2c_{n-1}\right) x^{n-1} = 0$$

Collect $\sum_{n=2}^{\infty} \left[(n-1)c_{n-1} - \left(3n c_n - 2c_{n-1}\right)\right] x^{n-1} - \left(3c_1 - 2c_0\right) = 0$

Simplify the bracts $\sum_{n=2}^{\infty} \left[(n+1)c_{n-1} - 3n c_n\right] x^{n-1} - \left(3c_1 - 2c_0\right) = 0$

Then $c_1 = \frac{2c_0}{3}$ *and* $(n+1)c_{n-1} - 3n c_n = 0$ $\therefore c_n = \frac{(n+1)c_{n-1}}{3n},\ n \ge 2$

at $n=2$ $c_2 = \frac{3c_1}{3.2} = \frac{3.2c_0}{3.2.3} = \frac{c_0}{3}$ *at* $n=3$ $c_3 = \frac{4c_2}{3.3} = \frac{4c_0}{3.3.3} = \frac{4c_0}{27}$

at $n=4$ $c_4 = \frac{5c_3}{3.4} = \frac{5}{3.4}.\frac{4c_0}{27} = \frac{5c_0}{81}$

The solution

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + ... + c_n x^n + ...$$

$$= c_0\left(1 + \frac{2}{3}x + \frac{1}{3}x^2 + \frac{4}{27}x^3 + \frac{5}{81}x^4 + ...\right)$$

(7) $y'' + xy' + y = 0$

Answer

We assume there is a solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$

We can differentiate the power series term by term, so

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Substitute in the equation $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$

Let the first series start from 0

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Separate first term from the first and third series to both start from 1

$$2a_2 + a_0 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} a_n x^n = 0$$

Now collect the series $2a_2 + a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n+1) a_n\right] x^n = 0$

Then we have

$$2a_2 + a_0 = 0 \rightarrow a_2 = -\frac{a_0}{2}$$

$$\left[(n+2)(n+1) a_{n+2} + (n+1) a_n\right], \; n = 1, 2, 3, \ldots$$

$$a_{n+2} = \frac{-(n+1)}{(n+1)(n+2)} a_n = \frac{-a_n}{(n+2)}, \; n = 1, 2, 3, \ldots \qquad (7)$$

We solve this recursion relation by putting successively in Equation 7

Put $n = 1$: $a_3 = \frac{-a_1}{3}$

Put $n = 2$: $a_4 = \frac{-1}{4} a_2 = \frac{-1}{4} a_2 = \frac{1}{2.4} a_0$

Put $n = 3$: $a_5 = \frac{-1}{5} a_3 = \frac{a_1}{3.5}$

Put $n = 4$: $a_6 = \frac{-1}{6} a_4 = \frac{-1}{2.4.6} a_0$

Put $n = 5$: $a_7 = \frac{-1}{7} a_5 = \frac{-a_1}{3.5.7}$

Put $n=6:$ $a_8=\frac{-1}{8}a_6=\frac{1}{2.4.6.8}a_0$

Put $n=7:$ $a_9=\frac{-1}{9}a_7=\frac{a_1}{3.5.7.9}$

In general, the even coefficients are given by $a_{2n}=\frac{(-1)^{n-1}}{2.4.6...(2n)}a_0=\frac{(-1)^{n-1}}{2^n(n)!}a_0$

And the odd coefficients are given by $a_{2n+1}=\frac{(-1)^n}{3.5.7...(2n+1)}a_1=\frac{(-1)^n 2^n\,\text{n!}}{(2n+1)!}a_1$

The solution is

$$y=a_0\left(1+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2^n(n)!}x^{2n}\right)+a_1\left(\sum_{n=0}^{\infty}\frac{(-1)^n 2^n\,\text{n!}}{(2n+1)!}x^{2n+1}\right)$$

---

(8) $y''=y$

Answer

We assume there is a solution of the form $y=\sum_{n=0}^{\infty}a_n x^n$

We can differentiate the power series term by term, so

$y'=\sum_{n=1}^{\infty}na_n x^{n-1}$ and $y''=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}$

Substitute in the equation then $\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}a_n x^n$

Let the first series start from 0 $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n=\sum_{n=0}^{\infty}a_n x^n$

Equating the coefficient in both sides $(n+2)(n+1)a_{n+2}=a_n$ and

$a_{n+2}=\frac{1}{(n+1)(n+2)}a_n,\ n=0,1,2,3,...$

We solve this recursion relation by putting successively in the equation.

Put $n=0:$ $a_2=\frac{a_0}{1.2}=\frac{a_0}{2!}$

Put $n=1:$ $a_3=\frac{1}{2.3}a_1=\frac{1}{3!}a_1$

Put $n=2:$ $a_4=\frac{1}{3.4}a_2=\frac{a_0}{1.2.3.4}=\frac{a_0}{4!}$

Put $n=3:$ $a_5=\frac{1}{4.5}a_3=\frac{1}{2.3.4.5}a_1=\frac{1}{5!}a_1$

Put $\quad n=4: \qquad a_6=\frac{1}{5.6}a_4=\frac{a_0}{1.2.3.4.5.6}=\frac{a_0}{6!}$

In general, the even coefficients are given by $a_{2n}=\frac{1}{(2n)!}a_0$

And the odd coefficients are given by $a_{2n+1}=\frac{1}{(2n+1)!}a_1$

The solution is

$$y=a_0\sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n}+a_1\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}x^{2n+1}$$

---

(9) $y''-xy=0$

<u>Answer</u>

Let $y=\sum_{n=0}^{\infty}c_n x^n$ Then $y'=\sum_{n=1}^{\infty}nc_n x^{n-1}$ and $y''=\sum_{n=2}^{\infty}n(n-1)c_n x^{n-2}$

By substitute in the equation we have

$$\sum_{n=2}^{\infty}n(n-1)c_n x^{n-2}-\sum_{n=0}^{\infty}c_n x^{n+1}=0$$

In the second series the power of $\boldsymbol{x}$ in the general term is $\boldsymbol{n+1}$ if we change it to becomes $\boldsymbol{n-2}$ then the summation start from 3 to unchanged the terms of the series.

$$\sum_{n=2}^{\infty}n(n-1)c_n x^{n-2}-\sum_{n=3}^{\infty}c_{n-3}x^{n-2}=0$$

Separate the first term from the first series

$$2.1c_2+\sum_{n=3}^{\infty}n(n-1)c_n x^{n-2}-\sum_{n=3}^{\infty}c_{n-3}x^{n-2}=0$$

Now the power of $\boldsymbol{x}$ in the general term in both series are equal and the summation start from 3 which leads to write the two terms as a one term as following.

$$2.1c_2+\sum_{n=3}^{\infty}\left[n(n-1)c_n-c_{n-3}\right]x^{n-2}=0$$

Compare the coefficients of x in both sides we have

$c_2=0$

$c_n=\frac{c_{n-3}}{n(n-1)},\ n\geq 3$ $c_n=\frac{c_{n-3}}{n(n-1)},\ n\geq 3$

$c_3 = \frac{c_0}{3.2} = \frac{c_0}{3!}$, $\quad c_4 = \frac{c_1}{4.3} = \frac{2c_1}{4.3.2} = \frac{2c_1}{4!}$

$c_5 = \frac{c_2}{20} = 0$, $\quad c_6 = \frac{c_3}{6.5} = \frac{c_0}{6.5.3} = \frac{4c_0}{6!}$

$c_7 = \frac{c_4}{7.8} = \frac{2c_1}{7.8.4!} = \frac{10c_1}{8!}$

Substitute by the coefficient in the hypothesis $y = \sum_{n=0}^{\infty} c_n x^n$

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + ...$$

$$= c_0 + c_1 x + 0 + \frac{c_0}{3!} x^3 + \frac{2c_1}{4!} x^4 + 0 + \frac{4c_0}{6!} x^6 + \frac{10c_1}{8!} x^7 + ...$$

$$= c_0 \left[1 + \frac{x^3}{3!} + \frac{4x^6}{6!} + ... \right] + c_1 \left[x + \frac{2x^4}{4!} + \frac{10x^7}{8!} + ... \right]$$

---

(10) $\quad y'' - xy' - y = 0, \quad y(0) = 1, y'(0) = 0$

Answer

We assume there is a solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$

We can differentiate the power series term by term, so

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Substitute in the equation $\quad \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$

Let the first series start from 0

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Separate first term from the first and third series to both start from 1

$$2a_2 - a_0 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} a_n x^n = 0$$

Now collect the series $\quad 2a_2 - a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - (n+1) a_n \right] x^n = 0$

Then we have $\quad 2a_2 - a_0 = 0 \;\rightarrow\; a_2 = \frac{a_0}{2}$

$\left[(n+2)(n+1)a_{n+2}-(n+1)a_n\right],\ n=1,2,3,...$

$$a_{n+2}=\frac{(n+1)}{(n+1)(n+2)}a_n=\frac{a_n}{(n+2)},\ n=1,2,3,... \qquad (7)$$

We solve this recursion relation by putting successively in Equation 7

Put $n=1:$ $a_3=\frac{a_1}{3}$

Put $n=2:$ $a_4=\frac{1}{4}a_2=\frac{1}{4}a_2=\frac{1}{2.4}a_0$

Put $n=3:$ $a_5=\frac{1}{5}a_3=\frac{a_1}{3.5}$

Put $n=4:$ $a_6=\frac{1}{6}a_4=\frac{1}{2.4.6}a_0$

Put $n=5:$ $a_7=\frac{1}{7}a_5=\frac{a_1}{3.5.7}$

Put $n=6:$ $a_8=\frac{1}{8}a_6=\frac{1}{2.4.6.8}a_0$

Put $n=7:$ $a_9=\frac{1}{9}a_7=\frac{a_1}{3.5.7.9}$

In general, the even coefficients are given by $a_{2n}=\frac{1}{2.4.6...(2n)}a_0=\frac{1}{2^n(n)!}a_0$

And the odd coefficients are given by $a_{2n+1}=\frac{1}{3.5.7...(2n+1)}a_1=\frac{2^n\,\text{n!}}{(2n+1)!}a_1$

The solution is

$$y=a_0\left(1+\sum_{n=1}^{\infty}\frac{1}{2^n(n)!}x^{2n}\right)+a_1\left(\sum_{n=0}^{\infty}\frac{2^n\,\text{n!}}{(2n+1)!}x^{2n+1}\right) \qquad (**)$$

When x=0 $a_0=1$

Differentiate with respect to x and substitute by x=0 then $a_1=0$ then the solution is

$$y=\sum_{n=0}^{\infty}\frac{x^{2n}}{2^n n!}$$

Use ratio test to discussed the divergence of the series

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{x^{2n+2}}{2^{n+1}(n+1)!}\frac{2^n n!}{x^{2n}}=\lim_{n\to\infty}\frac{x^2}{2(n+1)}=0$$

Then the series converges for all x

---

(11) $y'' + x^2 y = 0$

Answer

We assume there is a solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$ (1)

We differentiate power series term by term, so

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad (2)$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \quad (3)$$

In order to compare the expressions for and y'' more easily, we rewrite y'' as follows:"

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \quad (4)$$

Substituting the expressions in Equations 2 and 4 into the differential equation, we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Separate the two first terms from the first series

$$1.2a_2 + 2.3a_3 x + \sum_{n=2}^{\infty} (n+1)(n+2) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Let the first series start from 0

$$1.2a_2 + 2.3a_3 x + \sum_{n=0}^{\infty} (n+3)(n+4) a_{n+4} x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Collect

$$1.2a_2 + 2.3a_3 x + \sum_{n=0}^{\infty} \left[(n+3)(n+4) a_{n+4} + a_n\right] x^{n+2} = 0 \quad (5)$$

If two power series are equal, then the corresponding coefficients must be equal. Therefore, the coefficients of x^n in Equation 5 must be 0:

$a_2 = a_3 = 0$

$\left[(n+3)(n+4) a_{n+4} + a_n\right] = 0$

$$a_{n+4} = \frac{-a_n}{(n+3)(n+4)}, \quad n = 0,1,2,3,.... \quad (6)$$

Equation (6) is called a recursion relation. If c_0, c_1 are known, this equation allows us to determine the remaining coefficients recursively by putting $n = 0,1,2,3,...$ in succession.

when $n = 0$ $a_4 = \frac{-a_0}{3.4} = \frac{-a_0}{12}$ when $n = 1$ $a_5 = \frac{-a_1}{4.5} = \frac{-a_1}{20}$

when $n = 2$ $a_6 = \frac{-a_2}{6.5} = 0$ when $n = 3$ $a_7 = \frac{-a_3}{6.7} = 0$

when $n = 4$ $a_8 = \frac{-a_4}{7.8} = \frac{a_0}{672}$ when $n = 5$ $a_9 = \frac{-a_5}{8.9} = \frac{a_1}{1440}$

$$y = a_0 + a_1 x + \frac{-a_0}{12} x^4 + \frac{-a_1}{20} x^5 + \frac{a_0}{672} x^8 + \frac{a_1}{1440} x^9 + ...$$

$$\boxed{y = a_0\left(1 - \frac{1}{12}x^4 + \frac{1}{672} x^8 - ...\right) + a_1\left(x - \frac{1}{20}x^5 + \frac{1}{1440}x^9 + ...\right)}$$

---

(12) $y'' + x^2 y' + xy = 0,\ y(0) = 0,\ y'(0) = 1$

Answer

$x = 0$ is ordinary point because $p(0) = 1$ then the solution can be take the form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

Then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

By substitute in the equation we have

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

Separate two terms from the first series and one from the third

$$2c_2 + 6c_3 x + \sum_{n=4}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n+1} + c_0 x + \sum_{n=1}^{\infty} c_n x^{n+1} = 0$$

Let the first series start by 1

$$2c_2 + (6c_3 + c_0)x + \sum_{n=1}^{\infty} (n+3)(n+2) c_{n+3} x^{n+1} + \sum_{n=1}^{\infty} n c_n x^{n+1} + \sum_{n=1}^{\infty} c_n x^{n+1} = 0$$

Collect the series

$$2c_2+(6c_3+c_0)x+\sum_{n=1}^{\infty}\left[(n+3)(n+2)c_{n+3}+nc_n+c_n\right]x^{n+1}=0$$

Compare the coefficients

$c_2=0$ and $c_3=\dfrac{c_0}{6}$

$$(n+3)(n+2)c_{n+3}+nc_n+c_n=0$$

The recurrence relation is

$$c_{n+3}=\frac{-(n+1)}{(n+2)(n+3)}c_n\,,\quad n=1.,2,3,....$$

when $n=1$ $\quad c_4=\dfrac{-2}{3.4}c_1$

when $n=2$ $\quad c_5=\dfrac{-3}{4.5}c_2=0$

when $n=3$ $\quad c_6=\dfrac{-4}{5.6}c_3=\dfrac{-4}{5.6}\dfrac{c_0}{6}$

when $n=4$ $\quad c_7=\dfrac{-5}{6.7}c_4=\dfrac{2.5}{3.4.6.7}c_1$

when $n=5$ $\quad c_8=\dfrac{-6}{7.8}c_5=0$

when $n=6$ $\quad c_9=\dfrac{-7}{8.9}c_6=\dfrac{2.7}{3.5.6.8.9}c_0$

$$y=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+c_6x^6+c_7x^7+c_8x^8+c_9x^9+...$$

$$y=c_0+c_1x+0+\frac{1}{6}c_0x^3=\frac{1}{6}c_1x^4+0-\frac{1}{45}c_0x^6+\frac{5}{252}c_1x^7+0+\frac{7}{3240}c_0x^9+...$$

$$\boxed{y=c_0\left(1+\frac{1}{6}x^3-\frac{1}{45}x^6+\frac{7}{3240}x^9+...\right)+c_1\left(x-\frac{1}{6}x^4+\frac{5}{252}x^7+...\right)}$$

Initial condition tell us that $c_{0=0}$ and $c_{1=1}$ then the solution is

$$\boxed{y=c_1\left(x-\frac{1}{6}x^4+\frac{5}{252}x^7+...\right)}$$

Answer Exercises (2)

Obtain solution of the following differential equations in ascending power of x starting for what values of x the series is convergence

(1) $4xy'' + y' - y = 0$ **(1)**

Solution:

We assume there is a solution of the form $y = \sum_{n=0}^{\infty} a_n x^{s+n}$

We differentiate the power series term by term, so

$$\therefore \frac{dy}{dx} = \sum_{n=0}^{\infty} a_n (s+n) x^{s+n-1},$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} a_n (s+n)(n+s-1) x^{s+n-2}$$

Substitute in (1)

$$\sum_{n=0}^{\infty} 4a_n (s+n)(n+s-1) x^{s+n-1} + \sum_{n=0}^{\infty} a_n (s+n) x^{s+n-1} + \sum_{n=0}^{\infty} a_n x^{s+n} = 0$$

$$\sum_{n=0}^{\infty} \left[(s+n)(4n+4s-3)\right] a_n x^{s+n-1} + \sum_{n=0}^{\infty} a_n x^{s+n} = 0$$

Collect the series

$$\sum_{n=0}^{\infty} \left[(s+n)(4n+4s-3)\right] a_n x^{s+n-1} + \sum_{n=0}^{\infty} a_n x^{s+n} = 0$$

Separate the first term from the first series

$$\left[(s)(4s-3)\right] a_0 + \sum_{n=1}^{\infty} \left[(s+n)(4n+4s-3)\right] a_n x^{s+n-1} + \sum_{n=0}^{\infty} a_n x^{s+n} = 0$$

Let the first series start from 0

$$\left[(s)(4s-3)\right] a_0 + \sum_{n=0}^{\infty} \left[(s+n+1)(4n+4s+1)\right] a_{n+1} x^{s+n} + \sum_{n=0}^{\infty} a_n x^{s+n} = 0$$

Collect the series

$$\left[(s)(4s-3)\right] a_0 + \sum_{n=0}^{\infty} \left\{\left[(s+n+1)(4n+4s+1)\right] a_{n+1} + a_n\right\} x^{s+n} = 0$$

The indicial equation is $\left[(s)(4s-3)\right] = 0$

Which gives two roots are $s = 0, \quad s = \frac{3}{4}$

$$\left\{\left[(s+n+1)(4n+4s+1)\right] a_{n+1} + a_n\right\} = 0$$

$$a_{n+1} = \frac{a_n}{(s+n+1)(4n+4s+1)} \qquad (2)$$

When $s = 0$ in (2) then $a_{n+1} = \dfrac{a_n}{(n+1)(4n+1)}$, $n = 0,1,2,\ldots$ **and** $y = \sum_{n=0}^{\infty} a_n x^n$

$a_1 = a_0$, $\quad a_2 = \dfrac{a_1}{2.5} = \dfrac{a_0}{2.5}$, $\quad a_3 = \dfrac{a_2}{3.9} = \dfrac{a_0}{2.5.3.9}$

$a_4 = \dfrac{a_3}{4.13} = \dfrac{a_0}{2.5.3.9.4.13}$,

$a_5 = \dfrac{a_4}{5.17} = \dfrac{a_0}{2.5.3.9.4.13.5.17}$

$a_6 = \dfrac{a_5}{6.21} = \dfrac{a_0}{2.5.3.9.4.13.5.17.6.21}$

In general

$$a_n = \frac{a_0}{n!1.5.9\ldots(4n-3)}$$

Substitute in the series $y = \sum_{n=0}^{\infty} a_n x^{s+n}$

$$y_1 = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots + a_n x^n +$$

$$= a_0\left[1 + x + \frac{1}{2.5}x^2 + \frac{1}{2.5.3.9}x^3 + \frac{1}{2.5.3.9.4.13}x^4 + \ldots\right]$$

$$y_1 = \left[1 + \sum_{n=1}^{\infty} \frac{1}{n!\ \ 1.5.9\ldots(4n-3)} x^n\right]$$

at $s = \dfrac{3}{4}$ substitute in (2)

$a_{n+1} = \dfrac{a_n}{(4n+7)(n+1)}$, $n = 1.2.3\ldots,$ and $y = x^{3/4} \sum_{n=0}^{\infty} a_n x^n$,

Now

$a_1 = \dfrac{a_0}{7.1}$, $\quad a_2 = \dfrac{a_1}{11.2} = \dfrac{a_0}{7.1.11.2}$

$a_3 = \dfrac{a_2}{15.3} = \dfrac{a_0}{7.1.11.2.15.3}$

$a_4 = \dfrac{a_3}{19.4} = \dfrac{a_0}{7.1.11.2.15.3.19.4}$

In general $a_n = \dfrac{a_0}{n!7.11.15.19\ldots(4n+3)}$, $n = 1,2,3,\ldots$ and

$$y_2 = x^{3/4}\left[1+\sum_{n=1}^{\infty} \frac{a_0}{n!7.11.15.19...(4n+3)} x^n\right]$$

$$y = Ay_1 + By_2$$
$$= A\left[1+\sum_{n=1}^{\infty} \frac{1}{n!1.5.9...(4n-3)} x^n\right] + Bx^{3/4}\left[1+\sum_{n=1}^{\infty} \frac{1}{n!7.11.15.19...(4n+3)} x^n\right]$$

Or $$y = A\left[\sum_{n=0}^{\infty} \frac{1}{n!1.5.9...(4n-3)} x^n\right] + Bx^{3/4}\left[\sum_{n=0}^{\infty} \frac{1}{n!7.11.15.19...(4n+3)} x^n\right]$$

Another technique

We can find the solution as a function of x and s then substitute about the two values of s as following

$$a_{n+1} = \frac{a_n}{(s+n+1)(4n+4s+1)} \qquad (2)$$

when $n=0$ then $a_1 = \frac{a_0}{(s+1)(4s+1)}$

when $n=1$ then $a_2 = \frac{a_1}{(s+2)(4s+5)} = \frac{a_0}{(s+1)(s+2)(4s+1)(4s+5)}$

when $n=2$ then $a_3 = \frac{a_2}{(s+3)(4s+9)} = \frac{a_0}{(s+1)(s+2)(s+3)(4s+1)(4s+5)(4s-9)}$

in general $a_k = \frac{a_0}{(s+1)(s+2)(s+3)...(s+k)(4s+1)(4s+5)(4s+9)...(4s+4k-3)}$

$$z(x,s) = x^s \sum_{n=0}^{\infty} a_n x^n = x^s\left[a_0 + \sum_{k=0}^{\infty} a_k x^k\right]$$
$$= x^s\left[a_0 + \sum_{k=0}^{\infty} \frac{a_0}{(s+1)(s+2)(s+3)...(s+k)(4s+1)(4s+5)(4s+9)...(4s+4k-3)} x^k\right]$$

When s=0

$$y_1 = z(x,0) = \left[a_0 + \sum_{k=0}^{\infty} \frac{a_0}{(1)(2)(3)...(k).(1)(5)(9)...(4k-3)} x^k\right]$$
$$= a_0\left[1+\sum_{k=0}^{\infty} \frac{1}{k!\ (1)(5)(9)...(4k+3)} x^k\right]$$

When s=3/4

$$y_2 = z(x,\tfrac{3}{4}) = x^{\frac{3}{4}}\left[a_0 + \sum_{k=1}^{\infty} \frac{a_0}{(\frac{7}{4})(\frac{11}{4})(\frac{15}{4})\ldots\frac{(3+4k)}{4}(4)(8)(12)\ldots(4k)} x^k\right]$$

$$= x^{\frac{3}{4}}\left[a_0 + \sum_{k=1}^{\infty} \frac{a_0}{7.11.15\ldots(4k+3).k\,!} x^k\right]$$

$$y = Ay_1 + By_2$$

$$= A\left[1 + \sum_{n=1}^{\infty} \frac{1}{n\,!1.5.9\ldots(4n-3)} x^n\right] + Bx^{3/4}\left[1 + \sum_{n=1}^{\infty} \frac{1}{n\,!7.11.15.19\ldots(4n+3)} x^n\right]$$

---

(4) $9x(1-x)y'' - 12y' + 4y = 0$

Answer

We assume there is a solution of the form $y = \sum_{n=0}^{\infty} a_n x^{s+n}$

We differentiate power series term by term, so

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} a_n (s+n) x^{s+n-1}, \qquad \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} a_n (s+n)(n+s-1) x^{s+n-2}$$

Substitute in the differential equation then

$$\sum_{n=0}^{\infty} 9(s+n)(n+s-1)a_n x^{s+n-1} - \sum_{n=0}^{\infty} 9(s+n)(n+s-1)a_n x^{s+n}$$

$$-\sum_{n=0}^{\infty} 12a_n(s+n)x^{s+n-1} + \sum_{n=0}^{\infty} 4a_n x^{s+n} = 0$$

Collect the series

$$\sum_{n=0}^{\infty}\left[9(s+n)(n+s-1) - 12(s+n)\right]a_n x^{s+n-1} - \sum_{n=0}^{\infty}\left[9(s+n)(n+s-1) - 4\right]a_n x^{s+n} = 0$$

Simplify the bracts

$$\sum_{n=0}^{\infty}\left[(s+n)(9n+9s-21)\right]a_n x^{s+n-1} - \sum_{n=0}^{\infty}\left[(3s+3n+1)(3s+3n-4)\right]a_n x^{s+n} = 0$$

Separate the first term from first series

$$[(s)(9s-21)]a_0x^{s-1}$$

$$+\sum_{n=1}^{\infty}\ [(s+n)(9n+9s-21)]a_nx^{s+n-1}-\sum_{n=0}^{\infty}\ [(3s+3n+1)(3s+3n-4)]a_nx^{s+n}=0$$

let the first series start by 0

$$[(s)(9s-21)]a_0x^{s-1}$$

$$+\sum_{n=0}^{\infty}[(s+n+1)(9n+9s-12)]a_{n+1}x^{s+n}-\sum_{n=0}^{\infty}[(3s+3n+1)(3s+3n-4)]a_nx^{s+n}=0$$

Collect the series

$$[(s)(9s-21)]a_0x^{s-1}$$

$$+\sum_{n=0}^{\infty}\ \{(s+n+1)(9n+9s-12)a_{n+1}-(3s+3n+1)(3s+3n-4)a_n\}x^{s+n}=0$$

The indicial equation is

$[(s)(9s-21)]=0$ then $s=0,\ s=\frac{7}{3}$ and

$$(s+n+1)(9n+9s-12)a_{n+1}-(3s+3n+1)(3s+3n-4)a_n$$

$$a_{n+1}=\frac{(3s+3n+1)(3s+3n-4)}{3(s+n+1)(3n+3s-4)}a_n=\frac{(3s+3n+1)}{3(s+n+1)}a_n,\ n=0,1,2,...$$

Case (1) When $s=0$ then

$$a_{n+1}=\frac{(3n+1)}{3(n+1)}a_n,\ n=0,1,2,...\ \text{and}\ y=\sum_{n=0}^{\infty}\ a_nx^n$$

$$a_{n+1}=\frac{3n+1}{3(n+1)}a_n$$

at $n=0$ $\qquad a_1=\frac{1}{3}a_0$

at $n=1$ $\qquad a_2=\frac{4}{3.(2)}a_1=\frac{1.4}{3^2(2)!}a_0$

at $n=2$ $\quad a_3=\frac{7}{3.3}a_2=\frac{7}{3.3}\frac{1.4}{3^2(2)!}a_0=\frac{1.4.7}{3^3.3!}a_0$

at $n=3$ $\quad a_4=\frac{10}{3.4}a_3=\frac{10}{3.4}\frac{1.4.7}{3^3.3!}a_0=\frac{1.4.7.10}{3^4.4!}a_0$

$$a_n=\frac{1.4.7.10...(3n-2)}{3^n.n!}a_0$$

Then

$$y_1(x)=a_0+a_1x+a_2x^2+....+a_nx^n+...$$

$$=a_0\left[1+\frac{a_0}{3}x+\frac{1.4.}{3^2 2!}x^2+....+\frac{1.4.7....(3n-2)}{3^n n!}x^n+...\right]$$

$$=a_0\left[1+\sum_{n=1}^{\infty}\frac{1.4.7....(3n-2)}{3^n n!}x^n\right]$$

$$\boxed{y_1(x)=1+\sum_{n=1}^{\infty}\frac{1.4.7....(3n-2)}{3^n n!}x^n=(1-x)^{-1/3}}$$

When $s=\frac{7}{3}$ substitute in a_{n+1}

$$a_{n+1}=\frac{(3s+3n+1)}{3(s+n+1)}a_n,\ n=0,1,2,...$$

Then $a_{n+1}=\frac{(3n+8)}{(3n+10)}a_n,\ n=0,1,2,...$

at $n=0$ $\quad a_1=\frac{8}{10}a_0$

at $n=1$ $\quad a_2=\frac{11}{13}a_1=\frac{8.11}{10.13}a_0$

at $n=2$ $\quad a_3=\frac{14}{16}a_1=\frac{8.11.14}{10.13.16}a_0$

$$a_n=\frac{8.11.14...(3n+5)}{10.13.16...(3n+7)}a_0,\quad n=1,2,3,4,...$$

$$y_2 = x^s\left[a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...\right]$$

$$y_2 = x^{7/3}a_0\left[1 + \frac{8}{10}x + \frac{8.11}{10.13}x^2 + ... + \frac{8.11.14...(3n+5)}{10.13.16...(3n+7)}x^n + ...\right]$$

$$\boxed{y_2(x) = x^{7/3}\left\{1 + \sum_{n=1}^{\infty}\frac{8.11.14...(3n+5)}{10.13.16...(3n+7)}x^n\right\}}$$

General Solution

$Y = Ay_1 + By_2$ Where A and B are arbitrary constants?

Bessel Equation

In this section we consider three special cases of Bessel's equation,

$$x^2y'' + xy' + (x^2 - m^2)y = 0 \qquad (1)$$

Where m is a constant,. It is easy to show that $x = 0$ is a regular singular point. For simplicity we consider only the case $x > 0$

Bessel Equation of Order Zero.

This example illustrates the situation in which the roots of the indicial equation are equal. Setting $m = 0$ in Eq. (1) gives

$$x^2y'' + xy' + x^2y = 0$$

---

(1) The Bessel equation of order zero is $x^2y'' + xy' + x^2y = 0$ show that the roots of indicial equation are $s_1 = s_2 = 0$ and one solution for $x > 0$ is $J_0 = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ show that the series converges for all x .

Answer

Since $x = 0$ regular singular point then the solution in the form

$$z(x,s) = x^s\sum_{n=0}^{\infty}a_nx^n$$

Substitute in the equation then

$$\sum_{n=0}^{\infty}(n+s)(n+s-1)a_nx^{n+s} + \sum_{n=0}^{\infty}(n+s)a_nx^{n+s} + \sum_{n=0}^{\infty}a_nx^{n+s+2} = 0$$

Collect the first and the second series

$$\sum_{n=0}^{\infty}(n+s)^2a_nx^{n+s} + \sum_{n=0}^{\infty}a_nx^{n+s+2} = 0$$

Separate two terms from the first series we have

$$s^2 a_0 x^s + (1+s)^2 a_1 x^{s+1} + \sum_{n=0}^{\infty} (n+2+s)^2 a_{n+2} x^{n+s+2} + \sum_{n=0}^{\infty} a_n x^{n+s+2} = 0$$ Equating the coefficient in both sides

$$s^2 a_0 = 0$$

$$(1+s)^2 a_1 = 0$$

The recurrence relation is $(n+2+s)^2 a_{n+2} + a_n = 0$

Then $s = 0$ and $a_1 = 0$

$$a_{n+2} = \frac{-1}{(n+s+2)^2} a_n$$

$$a_2 = \frac{-1}{(s+2)^2} a_0$$

$$a_4 = \frac{-1}{(s+4)^2} a_2 = \frac{1}{(s+2)^2 (s+4)^2} a_0$$

$$a_6 = \frac{-1}{(s+6)^2} a_4 = \frac{-1}{(s+2)^2 (s+4)^2 (s+6)^2} a_0$$

$$a_{2k} = \frac{(-1)^k}{(s+2)^2 (s+4)^2 (s+6)^2 \ldots (s+2k)} a_0$$

Then

$$z(x,s) = a_0 x^s \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(s+2)^2 (s+4)^2 (s+6)^2 \ldots (s+2k)} x^{2k} \right]$$

Setting $s = 0$

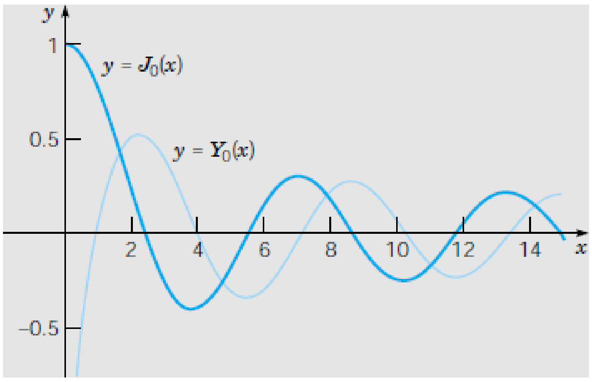
$$y_1 = z(x,0) = a_0 \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k} (k!)^2} x^{2k} \right]$$

The function in brackets is known as the Bessel function of the first kind of order zero and is denoted by $J_0(x)$. The series converges for all x, and that $J_0(x)$ is analytic at $x = 0$.

$$\boxed{J_0 = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k} (k!)^2} x^{2k}}$$

Note that $y_2 = \left[\frac{d}{ds} z(x,s) \right]_{s=0}$.

$$y_2 = J_0 \ln x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^{2k}(k!)^2} c_k x^{2k} \quad \text{where } c_k = \sum_{m=1}^{k} \frac{1}{m}$$



Simple solution

Since $x=0$ regular singular point then the solution in the form

$$y(x,s) = x^s \sum_{n=0}^{\infty} a_n x^n$$

Substitute in the equation then

$$\sum_{n=0}^{\infty} (n+s)(n+s-1)a_n x^{n+s} + \sum_{n=0}^{\infty} (n+s)a_n x^{n+s} + \sum_{n=0}^{\infty} a_n x^{n+s+2} = 0$$

Collect the first and the second series

$$\sum_{n=0}^{\infty} (n+s)^2 a_n x^{n+s} + \sum_{n=0}^{\infty} a_n x^{n+s+2} = 0$$

Separate two terms from the first series we have

$$s^2 a_0 x^s + (1+s)^2 a_1 x^{s+1} + \sum_{n=0}^{\infty} (n+2+s)^2 a_{n+2} x^{n+s+2} + \sum_{n=0}^{\infty} a_n x^{n+s+2} = 0$$ Equating the coefficient in both sides

$$s^2 a_0 = 0$$

$$(1+s)^2 a_1 = 0$$

The recurrence relation is $(n+2+s)^2 a_{n+2} + a_n = 0$

Then $s=0$ and $a_1 = 0$

$$a_{n+2} = \frac{-1}{(n+s+2)^2} a_n$$

Setting $s=0$

$$a_{n+2} = \frac{-1}{(n+2)^2} a_n$$

$$a_2 = \frac{-1}{(2)^2} a_0$$

$$a_4 = \frac{-1}{(4)^2} a_2 = \frac{1}{(2)^2(4)^2} a_0$$

$$a_6 = \frac{-1}{(6)^2} a_4 = \frac{-1}{(2)^2(4)^2(6)^2} a_0$$

$$a_{2k} = \frac{(-1)^k}{(2)^2(4)^2(6)^2 \ldots (2k)} a_0$$

Then

$$y(x,0) = a_0 \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2)^2(4)^2(6)^2 \ldots (2k)} x^{2k} \right]$$

$$= a_0 \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k}(1)^2(2)^2(3)^2 \ldots (k)} x^{2k} \right] = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k}(k!)} x^{2k}$$

This function is known as the Bessel function of the first kind of order zero and is denoted by $J_0(x)$. The series converges for all x, and that $J_0(x)$ is analytic at $x = 0$.

$$\boxed{J_0 = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2} \right)^{2k}}$$

Bessel Equation of Order One. This example illustrates the situation in which the roots of the indicial equation differ by a positive integer and the second solution involves a logarithmic term. Setting $m = 1$ in Bessel equation gives

$$x^2 y'' + xy' + (x^2 - 1)y = 0$$

(2) The Bessel equation of order one is $x^2 y'' + xy' + (x^2 - 1)y = 0$ show that the roots of indicial equation are $s_1 = 1, s_2 = -1$ and one solution for $x > 0$ is

$$J_1 = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n} n!(n+1)!}$$ show that the series converges for all x

Answer

Since $x = 0$ regular singular point then the solution in the form

$$z(x,s) = x^s \sum_{n=0}^{\infty} a_n x^n$$

Substitute in the equation then

$$\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s}+\sum_{n=0}^{\infty}(n+s)a_n x^{n+s}+\sum_{n=0}^{\infty}a_n x^{n+s+2}-\sum_{n=0}^{\infty}a_n x^{n+s}=0$$

Collect the series

$$\sum_{n=0}^{\infty}(n+s-1)(n+s+1)a_n x^{n+s}+\sum_{n=0}^{\infty}a_n x^{n+s+2}=0$$

Separate two terms from the first series we have

$$(s-1)(s+1)a_0 x^{s}+(s)(s+2)a_1 x^{s+1}$$

$$+\sum_{n=2}^{\infty}(n+s-1)(n+s+1)a_n\ x^{n+s}+\sum_{n=2}^{\infty}a_{n-2}\ x^{n+s}=0 \qquad (24)$$

Equating the coefficient in both sides

$(s-1)(s+1)a_0=0$

$s(s+2)a_1=0$

Then $s_1=1$ and $s_2=-1$

And the recurrence relation is $(n+s-1)(n+s+1)a_n+a_{n-2}=0$

$$a_n=\frac{-1}{(n+s-1)(n+s+1)}a_{n-2},\quad n\ge 2$$

Corresponding to the larger root *s* = 1 the recurrence relation becomes

$$a_n=\frac{-1}{n(n+2)}a_{n-2},\quad n\ge 2$$

We also find from the coefficient of x^{r+1} in Eq. (24) that $a_1=0$; hence from the recurrence relation $a_3=a_5=....=0$

. For even values of *n*, let *n* = 2*m*; then

$$a_{2m}=\frac{-1}{2m(2m+2)}a_{2m-2}=\frac{-1}{2^2 m(m+1)}a_{2m-2}\quad m=1,2,3,...$$

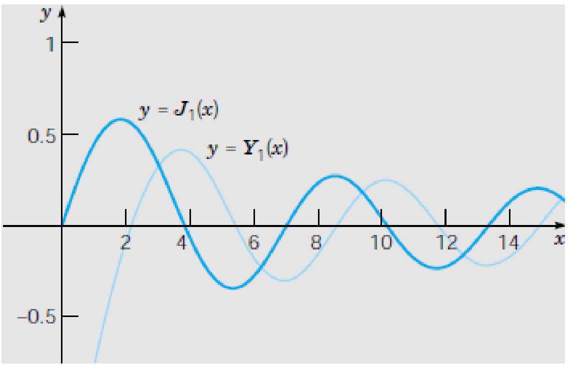
By solving this recurrence relation we obtain

$$a_{2m}=\frac{(-1)^m}{2^{2m}m!(m+1)!}a_0,\quad m=1,2,3,...$$

The Bessel function of the first kind of order one, denoted by J_1, is obtained by choosing $a_0=1/2$. Hence

$$J_1 = \sum_{m=0}^{\infty} \frac{(-1)^{m-1}}{(m)!(m+1)!}\left(\frac{x}{2}\right)^{2m+1}$$

The series converges absolutely for all x, so the function J_1 is analytic everywhere.



Bessel Equation of Order One-Half. This example illustrates the situation in which the roots of the indicial equation differ by a positive integer, but there is no logarithmic term in the second solution.

$$x^2y'' + xy' + (x^2 - \tfrac{1}{4})y = 0$$

Since $x = 0$ regular singular point then the solution in the form

$$z(x,s) = x^s \sum_{n=0}^{\infty} a_n x^n$$

Substitute in the equation then

$$\sum_{n=0}^{\infty} (n+s)(n+s-1)a_n x^{n+s} + \sum_{n=0}^{\infty} (n+s)a_n x^{n+s} + \sum_{n=0}^{\infty} a_n x^{n+s+2} - \sum_{n=0}^{\infty} \tfrac{1}{4}a_n x^{n+s} = 0$$

Collect the first and the second series

$$\sum_{n=0}^{\infty} \left[(n+s)^2 - \tfrac{1}{4}\right]a_n x^{n+s} + \sum_{n=0}^{\infty} a_n x^{n+s+2} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+s-\tfrac{1}{2})(n+s+\tfrac{1}{2})\right]a_n x^{n+s} + \sum_{n=0}^{\infty} a_n x^{n+s+2} = 0$$

Separate two terms from the first series we have

$$(s-\tfrac{1}{2})(s+\tfrac{1}{2})a_0 x^s + (s+\tfrac{1}{2})(s+\tfrac{3}{2})a_1 x^{s+1}$$

$$+ \sum_{n=2}^{\infty} \left[(n+s-\tfrac{1}{2})(n+s+\tfrac{1}{2})\right]a_n x^{n+s} + \sum_{n=2}^{\infty} a_{n-2} x^{n+s} = 0 \qquad (17)$$

Equating the coefficient in both sides

$(s-\frac{1}{2})(s+\frac{1}{2})a_0 = 0$

$(s+\frac{1}{2})(s+\frac{3}{2})a_1 = 0$

hence the roots are $s_1 = \frac{1}{2}$ and $s_2 = -\frac{1}{2}$ differ by an integer. The recurrence relation is

$\left[(n+s-\frac{1}{2})(n+s+\frac{1}{2})\right]a_n + a_{n-2} = 0$

$$a_n = -\frac{1}{(n+s-\frac{1}{2})(n+s+\frac{1}{2})}a_{n-2},\ n \geq 2 \qquad (18)$$

Corresponding to the larger root $s_1 = \frac{1}{2}$ we find from the coefficient of x^{s+1} that $a_1 = 0$. Hence, $a_3 = a_5 = ... = a_{2k+1} = ... = 0$

Further, for $s_1 = \frac{1}{2}$

$$a_n = -\frac{1}{(n)(n+1)}a_{n-2},\ n = 2,4,6,...$$

Setting $n = 2m$ then $a_{2m} = -\frac{1}{(2m)(2m+1)}a_{2m-2},\ n = 1,2,3,...$

By solving this recurrence relation we find that

$$a_{2m} = -\frac{1}{(2m)(2m+1)} = \frac{1}{2m(2m+1)(2m-2)(2\text{m}-3)}a_{2m-4}$$

$$= -\frac{1}{2m(2m+1)(2m-2)(2\text{m}-3)(2\text{m}-4)(2\text{m}-5)}a_{2m-6}$$

In general $a_{2m} = \frac{(-1)^m}{(2m+1)!}a_0,\ m = 1,2,3,....$

Hence taking $a_0 = 0$

$$y = x^{\frac{1}{2}}\left[1 + \sum_{m=1}^{\infty}\frac{(-1)^m}{(2m+1)!}x^{2m}\right] = x^{-\frac{1}{2}}\left[\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}x^{2m+1}\right] \qquad (20)$$

The power series in Eq. (20) is precisely the Taylor series for sin *x*; hence one solution of the Bessel equation of order one-half is $x^{-\frac{1}{2}}\sin x$. The Bessel function of the first kind of order one-half, $J_{\frac{1}{2}}$, is defined as $\left(\frac{2}{\pi}\right)^{\frac{1}{2}}y_1$. Thus

$$J_{\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\sin x,\ \ x > 0$$

Corresponding to the root $s_2 = -\frac{1}{2}$ it is possible that we may have difficulty in computing a_1 since $N = s_1 - s_2 = 1$. However, from Eq. (17) for $s_2 = -\frac{1}{2}$ the coefficients of x^s and x^{s+1} are both zero regardless of the choice of a_0 and a_1. Hence a_0 and a_1 can be chosen arbitrarily. From the recurrence relation (18) we obtain a set of even-numbered coefficients corresponding to a_0 and a set of odd-numbered coefficients corresponding to a_1. Thus no logarithmic term is needed to obtain a second solution in this case. It is left as an exercise to show that, for

$$s_2 = -\frac{1}{2}, \qquad a_{2n} = \frac{(-1)^n a_0}{(2n)!}, \qquad a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}, \quad n = 1, 2, 3, \ldots$$

Hence

$$y_2(x) = x^{-1/2}\left[a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} = \frac{(-1)^n}{(2n+1)!} x^{2n+1}\right]$$

$$= a_0 \frac{\cos x}{\sqrt{x}} + a_1 \frac{\sin x}{\sqrt{x}} \quad x > 0 \qquad (21)$$

The constant a_1 simply introduces a multiple of $y_1(x)$. The second linearly independent solution of the Bessel equation of order one-half is usually taken to be the solution for which $a_0 = \left(\frac{2}{\pi}\right)^{1/2}$ and $a_1 = 0$. It is denoted by $J_{-1/2}$. Then

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x, \qquad x > 0 \qquad (22)$$

The general solution is $y = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$ .The graphs of $J1/2$ and $J{-}1/2$ are shown in Figure